



Porter, R., & Evans, D. V. (2014). Connections between potentials describing wave scattering by complementary arrangements of vertical barriers. *Quarterly Journal of Mechanics and Applied Mathematics*, 67(2), 175-192. <https://doi.org/10.1093/qjmam/hbu002>

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CONNECTIONS BETWEEN POTENTIALS DESCRIBING WAVE SCATTERING BY COMPLEMENTARY ARRANGEMENTS OF VERTICAL BARRIERS.

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Summary

In this paper connections are made between the solutions of two water wave scattering problems, namely the diffraction of oblique waves by a thin vertical barrier with gaps and the complementary problem where the barriers are interchanged with the gaps. It is shown that the potential everywhere for the barrier problem is expressible in terms of the potential for the gap problem and a connection potential also associated with gaps in barriers. As a result the reflection coefficients are also shown to be connected.

The theory is illustrated in two ways. First, by analytically deriving Ursell's (1947) explicit result for a surface-piercing barrier in infinite depth from Dean's (1945) explicit result for a submerged barrier in infinite depth. Secondly, numerical results for complementary arrangements of barriers and gaps in finite depth and under oblique wave incidence are presented.

This paper is dedicated to the memory of Prof. Fritz Ursell who died in 2012.

1. Introduction

In the classical linearised theory of water waves only a few explicit solutions are known. One important class of problems having this feature involve thin vertical barriers in deep water where the fluid motion is two-dimensional. Thus, Dean (1) first solved the problem of time-harmonic waves normally-incident on a vertical barrier extending vertically downwards from a point below the free surface using complex variable methods. Shortly afterwards, Ursell (2) derived the solution to the geometrically complementary problem of a vertical barrier extending upwards through the free surface from a point below the free surface. Ursell (2) used a different method of solution to that of Dean (1), based on Havelock's (3) wavemaker theory involving integral transforms and was able to invert the integral equations that result. Ursell was also able to rederive the Dean solution using this approach.

In both Ursell and Dean problems, the reflection coefficients were shown to be expressible in terms of simple combinations of modified Bessel functions of argument $2\pi a/\lambda$, the only dimensionless parameter in the problem expressing the ratio of wavelength λ of surface waves to a the distance below the surface of the end of the barrier. Despite the similarity in these two expressions (repeated in this paper in equations (4.2) and (4.4)) there is no obvious connection between these two problems.

Later, various authors derived more complicated explicit solutions to vertical barrier problems in deep water. These included Lewin (4), Mei (5), Evans (6) and Porter (7) who considered scattering problems for finite barriers and finite gaps in barriers. Also of note is the work of Ursell (8) who considered radiation by forced oscillations of vertical barriers. Mandal and Chakrabati (9) provide an exhaustive catalogue of the work in this area and outline various methods that can be used to solve these and related problems.

When the fluid motion is not two-dimensional (for example when incident waves are obliquely incident upon vertical barriers), or when the fluid depth is no longer infinite, explicit solutions fail to exist and must be approximated. For example, Evans and Morris (10) ingeniously used the explicit solutions of Ursell (2) in a variational approximation to derive upper and lower bounds on the modulus of the reflection coefficients for oblique wave incidence upon surface piercing barriers in deep water.

For the most general case of finite depth and oblique incidence, solutions are most easily expressed in terms of eigenfunctions expansions. The most powerful application of this approach is outlined in Porter and Evans (11) who used matching of these separation solutions either side of an arrangement of vertical barriers and gaps to formulate integral equations for unknown functions related to physical quantities. A variational approach equivalent to the Galerkin method in which the unknown functions are expanded in a finite series of prescribed functions which incorporate physical properties of the fluid flow was shown to lead to accurate and efficient numerical results in addition to furnishing upper and lower bounds on various quantities such as the modulus of the reflection coefficient.

The purpose of the present paper is to consider wave scattering by vertical barriers with gaps and to connect those solutions with the solutions to the complementary problem in which the barriers and gaps are interchanged. For example, we shall demonstrate that it is possible to connect the reflection coefficients from Dean's (1) problem to Ursell's (2) problem.

The basis for this connection stems from an application of the ideas used in physics referred to as Babinet's principle (see Babinet (12)). Thus, in optics, it can be shown that wave scattering by a thin plane rigid screen with a hole can be related to a combination of solutions involving waves incident on a plane screen which occupies the region formerly being the hole. An application of Babinet's principle to the two-dimensional wave equation for oblique waves interacting with infinitely long barriers with gaps and the complementary problem of a finite length barrier is given in Linton and McIver (13). There it is shown that the solution of one problem for a particular incident wave angle is a combination of the solution to a complementary problem at the same wave angle and two additional solutions for waves incident at different angles. In the present paper these ideas are translated to a domain governed by Laplace's equation bounded by a free surface. The main difference here is that the extra potential(s) that is needed here to connect the two problems is defined in terms of a non-physical forcing, as opposed to waves incident from a different wave angle.

In section 2 we outline the governing equations for scattering of waves by vertical barriers. In section 3 it is shown how the velocity potentials describing the fluid motion for two complementary arrangements of barriers and gaps can be connected through the definition of a so-called connection potential. In Sections 4 and 5 we demonstrate how this theory works in two examples. In the first case it is shown explicitly how the Ursell potential can be constructed from the Dean potential through the derivation of a Dean-type connection potential. In the second case, we show how the methods of Porter and Evans (11) can be extended to connect solutions to two complementary barrier problems numerically.

2. Scattering of waves by vertical barriers

Cartesian coordinates are defined with the origin in the mean free surface and y pointing vertically downwards. The fluid is either of infinite depth or of constant finite depth, h . A thin barrier extending uniformly in the z direction occupies the interval $y \in B$ of the plane $x = 0$ and the gap in the barrier occupies the interval $y \in G$.

Plane monochromatic waves of radian frequency ω are incident from $x > 0$ on the barriers and propagate at an angle θ with respect to the plane $z = 0$.

Under the usual assumptions of inviscid linearised water wave theory the fluid motion can be described by a velocity potential which can be written as $\Re\{\Phi(x, y)e^{ilz}e^{-i\omega t}\}$ where $l = k \sin \theta$ and k is the wavenumber of the incident wave determined from the dispersion relation

$$\begin{aligned} \omega^2/g &\equiv K = k \tanh kh, & (\text{finite depth}) \\ K &= k, & (\text{infinite depth}). \end{aligned} \quad (2.1)$$

In the above g is gravitational acceleration. In the reduced two-dimensional setting, the complex-valued potential $\Phi(x, y)$ satisfies the equations

$$\Phi_{xx} + \Phi_{yy} - l^2\Phi = 0, \quad y > 0 \quad (2.2)$$

with

$$K\Phi + \Phi_y = 0, \quad \text{on } y = 0 \quad (2.3)$$

and

$$\begin{aligned} \Phi_y &= 0 \quad \text{on } y = h, & (\text{finite depth}) \\ |\nabla\Phi| &\rightarrow 0, \quad \text{as } y \rightarrow \infty, & (\text{infinite depth}). \end{aligned} \quad (2.4)$$

We must also impose no-flow conditions on the barrier

$$\Phi_x(0^\pm, y) = 0, \quad y \in B \quad (2.5)$$

and specify a radiation condition. We represent, with $\phi^0(x, y)$, a wave travelling in the positive x -direction; then $\phi^0(-x, y)$ is a wave travelling in the negative x -direction. We note that

$$\begin{aligned} \phi^0(x, y) &= e^{i\alpha x} \cosh k(y - h), & (\text{finite depth}) \\ \phi^0(x, y) &= e^{i\alpha x} e^{-Ky}, & (\text{infinite depth}) \end{aligned} \quad (2.6)$$

where $\alpha = k \cos \theta$ and for normal incidence $\alpha = k$.

For a wave incident from $x = \infty$

$$\Phi(x, y) \sim \begin{cases} \phi^0(-x, y) + R\phi^0(x, y), & x \rightarrow \infty \\ T\phi^0(-x, y), & x \rightarrow -\infty \end{cases} \quad (2.7)$$

where R and T are the reflection and transmission coefficients. It can easily be shown that

$$\Phi(x, y) = \begin{cases} \phi^0(x, y) + \phi^0(-x, y) + \phi(x, y), & x > 0 \\ -\phi(-x, y), & x < 0 \end{cases} \quad (2.8)$$

where $\phi(x, y)$ is defined in $x > 0$ and satisfies (2.2), (2.3) and (2.4) in addition to the boundary conditions

$$\phi_x(0, y) = 0, \quad y \in B, \quad \phi(0, y) + \phi^0(0, y) = 0, \quad y \in G \quad (2.9)$$

which result from imposing (2.5) on Φ and continuity of $\Phi(x, y)$ across $x = 0$ when $y \in G$, based on the decomposition (2.8). Under this definition

$$\phi(x, y) \sim (R - 1)\phi^0(x, y), \quad \text{as } x \rightarrow \infty \quad (2.10)$$

with $T = R - 1$.

With r a local measure of the distance from any barrier edge immersed in the fluid, we also require

$$|\nabla\phi| \sim O(r^{-1/2}), \quad \text{as } r \rightarrow 0. \quad (2.11)$$

3. Complementary problems and connection potentials

We consider two separate problems and let $\phi \equiv \phi^u$ represent a potential for a barrier occupying $y \in B_u$ and a gap in the barrier occupying $y \in G_u$. In a second problem, $\phi \equiv \phi^d$ for a barrier occupying $y \in B_d = G_u$ and a gap occupying $y \in G_d = B_u$. Thus, in the two problems the positions of the barriers and the gaps are reversed. Associated with each of these two problems, reflection coefficients, R , are labelled R^u and R^d .

It is the purpose in what follows to connect $\phi^{u/d}$ with $\phi^{d/u}$ and $R^{u/d}$ with $R^{d/u}$ (the notation $X^{u/d}$ implying X^u or X^d).

We will imagine that the u -problem represents a barrier directed *up* through the free surface from a point $(0, a)$ below the surface and the d -problem represents a barrier directed *down* to the bottom of the fluid from the same point $(0, a)$ below the surface. The methods described herein can be extended to more complex complementary arrangements of barriers.

As a way of motivating the need for a connection potential, consider, for example, the function ϕ_x^d which satisfies Laplace's equation, the conditions on the free surface and at the bottom of the fluid and radiates waves. We also note that on $y \in B^d$, $\phi_x^d = 0$, and hence ϕ_x^d satisfies all but one of the properties required of the complementary potential $\phi^u + \phi^0$ (since $B^d = G^u$). However, the x -derivative of ϕ_x^d on G^u is *not* proportional to $\phi_x^0(x, 0)$ and hence we add to the function ϕ_x^d , the x -derivative of what we call a connection potential, which satisfies all of the properties of ϕ^d apart from on G^u upon which a boundary condition will be set by the requirement that $\phi_x^u = 0$.

We relate the potentials ϕ^u and ϕ^d through the introduction of new potentials ψ^u and ψ^d satisfying (2.2), (2.3) and (2.4). This is done by writing

$$\phi^{u/d}(x, y) + \phi^0(x, y) = i\alpha^{-1} \left(\phi_x^{d/u}(x, y) + A^{d/u} \psi_x^{d/u}(x, y) \right) \quad (3.1)$$

where $A^{d/u}$ is a constant. Then on $y \in G_{u/d}$, the left-hand side of (3.1) is zero according to (2.9) whilst the first term on the right-hand side of (3.1) is also zero from (2.9) since $G_{u/d} = B_{d/u}$. So it must be that

$$\psi_x^{d/u}(0, y) = 0, \quad \text{on } y \in B_{d/u}. \quad (3.2)$$

Now we take the x -derivative of (3.1) and use the governing equation (2.2) before setting $x = 0$ to write

$$\phi_x^{u/d}(0, y) + \phi_x^0(0, y) = -i\alpha^{-1} \left(\frac{d^2}{dy^2} - l^2 \right) \left(\phi^{d/u}(0, y) + A^{d/u} \psi^{d/u}(0, y) \right). \quad (3.3)$$

We consider this equation on $y \in B_{u/d} = G_{d/u}$ where $\phi_x^{u/d}(0, y) = 0$ and $\phi^{d/u}(0, y) = -\phi^0(0, y)$ from (2.9). The definition of ϕ^0 given in (2.6) and satisfying (2.2) means that

$$\left(\frac{d^2}{dy^2} - l^2 \right) \phi^0(0, y) = -i\alpha \phi_x^0(0, y) \quad (3.4)$$

and it follows from (3.3) that

$$\left(\frac{d^2}{dy^2} - l^2 \right) \psi^{d/u}(0, y) = 0, \quad y \in G_{d/u}. \quad (3.5)$$

This second order differential equation may be integrated up and, assuming (as we have) that $G_{d/u}$

intersects with either the top or the bottom of the fluid, boundary conditions (2.3) or (2.4) apply to and thus we denote its general solution as

$$\psi^{d/u}(0, y) = f^{d/u}(y), \quad y \in G_{d/u} \quad (3.6)$$

where the function $f^{d/u}(y)$ will be specified later according to the particular problem being considered. An arbitrary integration constant is not included in the definition of $f^{d/u}(y)$ since it is accounted for by the constant $A^{d/u}$ in (3.1). This constant is determined by applying the condition

$$\lim_{r \rightarrow 0} r^{1/2} \left(\phi_x^{d/u}(x, y) + A^{d/u} \psi_x^{d/u}(x, y) \right) = 0, \quad \text{where } r = (x^2 + (y - a)^2)^{1/2} \quad (3.7)$$

which follows from (3.1) since $\phi^{u/d}(x, y)$ is bounded as $r \rightarrow 0$.

We have shown in (3.1) that $\phi^{u/d}$ can be expressed in terms of the sum of the x -derivative of $\phi^{d/u}$ and a ‘connection’ potential $\psi^{d/u}(x, y)$ satisfying the same homogeneous Neumann condition (3.2) as $\phi^{d/u}(x, y)$ on the barrier but with different Dirichlet conditions described by (3.6). The problem for $\psi^{d/u}(x, y)$ describes a potential in which waves are radiated to infinity and so we write

$$\psi^{u/d}(x, y) \sim \tilde{R}^{u/d} \phi^0(x, y), \quad x \rightarrow \infty \quad (3.8)$$

where $\tilde{R}^{u/d} \in \mathbb{C}$ is a radiated wave amplitude to be determined.

Using the far-field asymptotic form designated to each term in (3.1) and letting $x \rightarrow \infty$ gives

$$R^{u/d} = 1 - R^{d/u} - A^{d/u} \tilde{R}^{d/u} \quad (3.9)$$

and this demonstrates that the reflection coefficients from one problem may be expressed in terms of the complementary problem. We note in passing that repeated use of (3.9) implies that

$$A^d \tilde{R}^d = A^u \tilde{R}^u \quad (3.10)$$

although this relation appears to have no particular practical importance.

4. Calculations for infinite depth and normal incidence

As an example of the theory, we shall make a connection between two complementary problems in infinite depth and under normal wave incidence for which explicit solutions exist. Here, from (2.6), $\phi^0(x, y) = e^{iKx} e^{-Ky}$, $l = 0$ and $\alpha = k = K$.

In this example, ϕ^u will refer to the Ursell (2) barrier potential for a surface-piercing barrier submerged to a depth a below the surface in which $B_u = (0, a)$ and $G_u = (a, \infty)$. Thus, from Ursell (2) we have

$$\phi^u(x, y) + \phi^0(x, y) = C \left(\pi I_1(Ka) e^{iKx - Ky} + \int_0^\infty \frac{L(k, y) J_1(ka) e^{-kx}}{(k^2 + K^2)} dk \right) \quad (4.1)$$

using standard notation for Bessel functions and where $L(k, y) = k \cos ky - K \sin ky$ and $C = (\pi I_1(Ka) + iK_1(Ka))^{-1}$. Then

$$R^u = \pi I_1(Ka) C = \frac{\pi I_1(Ka)}{\pi I_1(Ka) + iK_1(Ka)}. \quad (4.2)$$

Next, ϕ^d refers to the Dean **(1)** potential for a barrier extending from the depths to a point a below the surface in which $B_d = (a, \infty)$ and $G_d = (0, a)$. Ursell **(2)** redrived Dean's potential, expressible as

$$\phi^d(x, y) + \phi^0(x, y) = B \left(K_0(Ka) e^{iKx - Ky} - \int_0^\infty \frac{L(k, y) J_0(ka) e^{-kx}}{(k^2 + K^2)} dk \right) \quad (4.3)$$

where $B = (K_0(Ka) + i\pi I_0(Ka))^{-1}$ and

$$R^d = K_0(Ka)B = \frac{K_0(Ka)}{K_0(Ka) + i\pi I_0(Ka)}. \quad (4.4)$$

For the connection potential, solving (3.5) with $l = 0$, gives (3.6) which, taking into account the conditions (2.3) and (2.4), is

$$\psi^d(0, y) = 1 - Ky, \quad y \in G_d, \quad \text{and} \quad \psi^u(0, y) = 1, \quad y \in G_u. \quad (4.5)$$

We shall derive now the connection potential $\psi^d(x, y)$ from first principles and confirm that together with the knowledge of the Dean potential (4.3) and the corresponding reflection coefficient (4.4) it can be used to derive the Ursell potential (4.1) and its reflection coefficient (4.2) using the relations (3.1) and (3.9). For this problem, $G_d = (0, a)$ and $B_d = (a, \infty)$. The methods outlined below for the calculation of ψ^d are similar to those used by Ursell **(2)** to find ϕ^d .

The most general potential satisfying (2.2), (2.3), (2.4) and (3.8) is written, using the integral transform of Havelock **(3)**,

$$\psi^d(x, y) = \tilde{R}^d e^{iKx - Ky} + \frac{2}{\pi} \int_0^\infty \frac{A^d(k) L(k, y) e^{-kx}}{k(k^2 + K^2)} dk, \quad (4.6)$$

where \tilde{R}^d and $A^d(k)$ are unknowns. We define

$$U^d(y) \equiv \psi_x^d(0, y) = iK \tilde{R}^d e^{-Ky} - \frac{2}{\pi} \int_0^\infty \frac{A^d(k) L(k, y)}{(k^2 + K^2)} dk \quad (4.7)$$

which is zero when $y \in B_d$ ($y > a$) on account of (3.2). Using Havelock's **(3)** inversion theorem

$$\tilde{R}^d = -2i \int_0^a U^d(y) e^{-Ky} dy, \quad \text{and} \quad A^d(k) = - \int_0^a U^d(y) L(k, y) dy \quad (4.8)$$

where use has been made of $U^d(y) = 0$ for $y > a$ to restrict the integration interval to $(0, a)$. It follows from substitution of $A^d(k)$ from (4.8) into (4.6) and the imposition of (4.5) that

$$\int_0^a U^d(t) K(y, t) dt = g(y), \quad y \in (0, a) \quad (4.9)$$

where

$$K(y, t) = \frac{1}{\pi} \int_0^\infty \frac{L(k, t) L(k, y)}{k(k^2 + K^2)} dk \quad (4.10)$$

and with

$$g(y) = \frac{1}{2} \left(\tilde{R}^d e^{-Ky} + Ky - 1 \right). \quad (4.11)$$

Following the method used by Ursell (2) for the scattering problem this integral equation may be transformed by first defining the differential operators

$$\mathcal{D}_y^\pm \equiv \frac{d}{dy} \pm K \quad (4.12)$$

which allows us to write $L(k, y) = \mathcal{D}_y^- (\sin ky)$. It follows from the definition (4.10) that

$$K(y, t) = \mathcal{D}_y^- \mathcal{D}_t^- \left(\frac{1}{\pi} \int_0^\infty \frac{\sin ky \sin kt}{k(k^2 + K^2)} dk \right) \quad (4.13)$$

and hence

$$\mathcal{D}_y^+ (K(y, t)) = -\mathcal{D}_t^- \left(\frac{1}{\pi} \int_0^\infty \frac{\sin ky \sin kt}{k} dk \right) = -\frac{1}{2\pi} \mathcal{D}_t^- \left(\ln \left| \frac{y+t}{y-t} \right| \right) \quad (4.14)$$

(see Gradshteyn and Ryzhik (14, §3.741, equation 1)).

Thus, for $y \in (0, a)$ we have

$$\begin{aligned} \mathcal{D}_y^+ \left(\int_0^a U^d(t) K(y, t) dt \right) &= -\frac{1}{2\pi} \int_0^a U^d(t) \mathcal{D}_t^- \left(\ln \left| \frac{y+t}{y-t} \right| \right) dt \\ &= \frac{K}{2\pi} \left[\ln \left| \frac{y+t}{y-t} \right| \int_a^t U^d(s) ds \right]_0^a - \frac{1}{2\pi} \int_0^a V^d(t) \frac{d}{dt} \ln \left| \frac{y+t}{y-t} \right| dt \end{aligned} \quad (4.15)$$

after integration by parts, where we have defined

$$V^d(y) = U^d(y) + K \int_a^y U^d(t) dt. \quad (4.16)$$

The first term on the right-hand side in (4.15) evaluates to zero. Notice that $\lim_{y \rightarrow a} (U^d(y) - V^d(y)) = 0$ and that $V^d(y)$ has the same singular behaviour as $U^d(y)$ near $y = a$ and is bounded near $y = 0$.

It follows from (4.15) and (4.9) that

$$\mathcal{D}_y^+ \left(\int_0^a U^d(t) K(y, t) dt \right) = -\frac{1}{2\pi} \int_0^a \frac{2yV^d(t)}{y^2 - t^2} dt = \mathcal{D}_y^+ (g(y)). \quad (4.17)$$

Thus

$$\int_0^a \frac{V^d(t)}{y^2 - t^2} dt = -\pi(g'(y) + Kg(y))/y, \quad y \in (0, a). \quad (4.18)$$

For the particular $g(y)$ given by (4.11) in this case $-(g'(y) + Kg(y))/y = -\frac{1}{2}K^2$ so that $V^d(y)$ satisfies

$$\int_0^a \frac{V^d(t)}{y^2 - t^2} dt = -\frac{1}{2}\pi K^2, \quad y \in (0, a). \quad (4.19)$$

Equation (4.19) is a special case of the general integration equation

$$\int_0^a \frac{\mu(t)}{y^2 - t^2} dt = \lambda(y), \quad y \in (0, a) \quad (4.20)$$

whose solution, for suitable $\lambda(y)$, and where $(a^2 - t^2)^{1/2}\mu(t)$ is bounded near $t = a$, can be shown, using a simple change of variables in a similar result in Ursell (2) applied to the interval (a, ∞) instead of $(0, a)$, to be

$$\mu(t) = \frac{1}{(a^2 - t^2)^{1/2}} \left(D + \frac{4}{\pi^2} \int_0^a \frac{\lambda(y)y^2(a^2 - y^2)^{1/2}}{y^2 - t^2} dy \right), \quad y \in (0, a) \quad (4.21)$$

where D is an arbitrary constant. Application of this general inversion formula to (4.19) gives

$$V^d(t) = \frac{1}{(a^2 - t^2)^{1/2}} \left(D - \frac{2K^2}{\pi} \int_0^a \frac{y^2(a^2 - y^2)^{1/2}}{y^2 - t^2} dy \right), \quad y \in (0, a) \quad (4.22)$$

and elementary integration results in

$$V^d(t) = \frac{D}{(a^2 - t^2)^{1/2}} - K^2(a^2 - t^2)^{1/2}, \quad t \in (0, a). \quad (4.23)$$

The constant, D , in the solution (4.23) originates from the transformation of the original integral equation (4.9) into (4.18) through the use of the differential operator \mathcal{D}_y^+ in (4.17). Some work is now needed to eliminate this constant.

First we make use of an integral identity between $U^d(t)$ and $V^d(t)$, which is easily established from the relation (4.16) and integration by parts, to obtain

$$\int_0^a L(k, t) U^d(t) dt = k \int_0^a V^d(t) \cos kt dt = \frac{1}{2} \pi (kD J_0(ka) - K^2 a J_1(ka)). \quad (4.24)$$

The final step in the above follows after using (4.23) and standard integral identities

$$\int_0^a \frac{\cos(ky)}{(a^2 - y^2)^{1/2}} dy = -\frac{\pi J_0(ka)}{2} \quad \text{and} \quad \int_0^a (a^2 - y^2)^{1/2} \cos(ky) dy = \frac{\pi a J_1(ka)}{2k}. \quad (4.25)$$

(e.g. McLachlan (15, p.202 equation 177)). It follows from using (4.24) in (4.9)–(4.11) that

$$\tilde{R}^d e^{-Ky} + Ky - 1 = D \int_0^\infty \frac{J_0(ka) L(k, y)}{(k^2 + K^2)} dk - K^2 a \int_0^\infty \frac{J_1(ka) L(k, y)}{k(k^2 + K^2)} dk, \quad y \in (0, a). \quad (4.26)$$

With some effort (see Appendix A), further integral relations can be established. In particular from (A.2)

$$\int_0^\infty \frac{J_0(ka) L(k, y)}{(k^2 + K^2)} dk = e^{-Ky} K_0(Ka) \quad (4.27)$$

and from (A.5)

$$\int_0^\infty \frac{J_1(ka) L(k, y)}{k(k^2 + K^2)} dk = \frac{(1 - Ky)}{K^2 a} - \frac{K_1(Ka) e^{-Ky}}{K}. \quad (4.28)$$

Substituting these into (4.26) we find that the terms $1 - Ky$ on each side of the equation cancel to leave

$$\tilde{R}^d = DK_0(Ka) + KaK_1(Ka) \quad (4.29)$$

which determines D (in terms of \tilde{R}^d). An expression for \tilde{R}^d follows from the first equation in (4.8) which can be written using the relation between U^d and V^d in (4.16) and integration by parts as

$$\tilde{R}^d = -2i \int_0^a \cosh(Ky) V^d(y) dy = -i\pi (DI_0(Ka) - KaI_1(Ka)) \quad (4.30)$$

and the final step comes from substitution of (4.23) and use of the results (4.25) with k replaced by iK .

Equations (4.29) and (4.30) may be combined to give

$$\tilde{R}^d = i\pi Ka (I_0(Ka)K_1(Ka) + I_1(Ka)K_0(Ka)) B, \quad \text{and} \quad D = i \frac{BKa}{C}, \quad (4.31)$$

where B and C are factors defined following (4.1) and (4.3).

We are almost in a position to determine R^u from (3.9) but first need to calculate A^d from (3.7). It can be shown (see Ursell (2)), en route to the derivation of the Dean potential ϕ^d , that

$$\phi_x^d(0, y) \sim \frac{B}{(a^2 - y^2)^{1/2}}, \quad \text{as } y \rightarrow a. \quad (4.32)$$

It follows from the comments that follow (4.16) and from (4.23) and the relation $U^d(y) \equiv \psi_x^d(0, y)$ that

$$\psi_x^d(0, y) \sim \frac{D}{(a^2 - y^2)^{1/2}}, \quad \text{as } y \rightarrow a. \quad (4.33)$$

Thus, from (3.7) we require $A^d D + B = 0$ and so (3.9) becomes

$$\begin{aligned} R^u &= 1 - R^d + B\tilde{R}^d/D \\ &= B(i\pi I_0(Ka) + \tilde{R}^d/D) \\ &= B\pi (iI_0(Ka) + C(I_0(Ka)K_1(Ka) + I_1(Ka)K_0(Ka))) = \pi I_1(Ka)C \end{aligned} \quad (4.34)$$

using (4.4), (4.31) and the definitions of C and B . This is the Ursell result, (4.2).

To derive the Ursell potential, ϕ^u , from (3.1) we first use (4.3) and (4.6), (4.8) and (4.24) to show that

$$\phi^d(x, y) + A^d \psi^d(x, y) = \left(BK_0(Ka) + A^d \tilde{R}^d - 1 \right) e^{iKx - Ky} + K^2 a A^d \int_0^\infty \frac{L(k, y) J_1(ka) e^{-kx}}{k(k^2 + K^2)} dk \quad (4.35)$$

where the resulting integral involving $J_0(ka)$ vanishes since $A^d D + B = 0$. It can be shown $BK_0(Ka) + A^d \tilde{R}^d = iCK_1(Ka)$ so that, from (4.3),

$$\begin{aligned} \phi^u(x, y) + e^{iKx - Ky} &= iK^{-1} \frac{\partial}{\partial x} (\phi^d(x, y) + A^d \psi^d(x, y)) \\ &= (1 - iCK_1(Ka)) e^{iKx - Ky} - iKa A^d \int_0^\infty \frac{L(k, y) J_1(ka) e^{-kx}}{(k^2 + K^2)} dk \\ &= C \left(\pi I_1(Ka) e^{iKx - Ky} + \int_0^\infty \frac{L(k, y) J_1(ka) e^{-kx}}{(k^2 + K^2)} dk \right) \end{aligned} \quad (4.36)$$

since $-iKaA^d = C$. This is precisely the Ursell potential given by (4.1).

4.1 Remarks

In the example illustrated above, the Ursell potential and the corresponding reflection coefficient for a surface-piercing barrier have been derived from the Dean potential and a connection potential defined by a Dean-type problem which is solved using methods which apply to the solution to Dean's problem for a barrier extending downwards to infinity.

We could equally have derived the connection potential $\psi^u(x, y)$ and used it in conjunction with the Ursell potential to derive the Dean potential. This has been confirmed by the authors but has not been included in this paper as it involves lengthy calculations similar to those already presented.

The particular solution method used here focuses on inverting an integral equation for an unknown function $U^d(y)$ for $y \in G_d$, relating to the horizontal component of the velocity across the gap above the submerged barrier. It is interesting to note that this is not the only approach that can be used for solving the Dean problem and the related Dean connection potential problem. An alternative approach involves formulating integral equations for the unknown potentials $\phi^d(0, y)$ and $\psi^d(0, y)$ along $y \in B_d$, the length of the barrier. Thus, the authors have also solved these integral equations and used the solutions that result to connect Dean and Ursell solutions via connection potentials. Again the details are lengthy and have not been included here. However, we shall return to this comment in Section 5.2.

5. Finite depth calculations

In this section we outline how the connection potentials and relations between reflection coefficients apply when the solution cannot be solved exactly and relies ultimately upon numerical solutions. In what follows, we adopt and then develop the methods of Porter and Evans (11). It helps to first outline the solution method for the scattering potentials before considering the connection potential.

A propagating wave in finite depth referred to in (2.6), is written

$$\phi^0(x, y) = e^{i\alpha x} \varphi_0(y) \quad (5.1)$$

where $\alpha^2 = k^2 - l^2 = k \cos \theta$ in terms of the incident wave angle θ and $\varphi_0(y) = \cosh k(h - y)$ where k satisfies (2.1). We also define

$$\varphi_n(y) = \cos k_n(h - y), \quad n \geq 1 \quad (5.2)$$

where k_n are the positive real roots of $K = -k_n \tan k_n h$. The set of eigenfunctions $\{\varphi_0(y), \varphi_1(y) \dots\}$, are complete in the space $L_2(0, h)$ and satisfy the orthogonality condition

$$\int_0^h \varphi_n(y) \varphi_m(y) dy = N_n h \delta_{mn}, \quad \text{where } N_n = \frac{1}{2} \left(1 + \frac{\sin 2k_n h}{2k_n h} \right) \quad (5.3)$$

and δ_{mn} is the Kronecker delta. We can extend the formulae given for φ_n and N_n when $n \geq 1$ to $n = 0$ by defining $k_0 = -ik$.

The potential ϕ^u and ϕ^d for each of the two complementary problems in which the barrier occupies $B_u = G_d$ and $B_d = G_u$ is expanded in separation solutions, thus

$$\phi^{u/d}(x, y) = (R^{u/d} - 1)e^{i\alpha x} \varphi_0(y) + \sum_{n=1}^{\infty} A_n^{u/d} e^{-\alpha_n x} \varphi_n(y), \quad x > 0, 0 < y < h \quad (5.4)$$

so that $\phi^{u/d}(x, y)$ satisfies (2.2), (2.3) and (2.4) and $R^{u/d}$ is the reflection coefficient in alignment

with the definition (2.10). Also, $A_n^{u/d}$ are expansion coefficients and $\alpha_n^2 = k_n^2 + l^2$. Thus, the expansions in (5.4) satisfy all the conditions of the problem apart from those on $x = 0$. We continue by defining

$$R^{u/d}U^{u/d}(y) \equiv \phi_x^{u/d}(0, y) = i\alpha(R^{u/d} - 1)\varphi_0(y) - \sum_{n=1}^{\infty} \alpha_n A_n^{u/d} \varphi_n(y). \quad (5.5)$$

Applying the first condition in (2.9) to (5.5) and using the orthogonality condition (5.3) gives

$$A_n^{u/d} = -\frac{R^{u/d}}{\alpha_n h N_n} \int_{G_{u/d}} U^{u/d}(y) \varphi_n(y) dy \quad (5.6)$$

for $n \geq 1$ and

$$i(R^{u/d} - 1) = R^{u/d} \mathcal{B}^{u/d}, \quad \text{where} \quad \mathcal{B}^{u/d} = \frac{1}{\alpha h N_0} \int_{G_{u/d}} U^{u/d}(y) \varphi_0(y) dy. \quad (5.7)$$

Next, taking (5.4) with $x = 0$, substituting (5.6) and imposing the second condition in (2.10) results in

$$\int_{G_{u/d}} U^{u/d}(t) K(y, t) dt = \varphi_0(y), \quad y \in G_{u/d} \quad (5.8)$$

where, here,

$$K(y, t) = \sum_{n=1}^{\infty} \frac{\varphi_n(y) \varphi_n(t)}{\alpha_n h N_n}. \quad (5.9)$$

Once $U^{u/d}(y)$ is determined from (5.8), $R^{u/d}$ can be calculated from (5.7).

Unlike the case of infinite depth and normal incidence considered in Section 4, the integral equation (5.8) cannot be inverted explicitly. Numerical solutions to (5.8) based on an accurate Galerkin approximation are outlined in Porter and Evans (11). A summary of the procedure is provided in Appendix B, accompanied with additional details associated with the connection potential which follows.

The relations (3.1) and (3.9) allow us to express $\phi^{u/d}(x, y)$ in terms of $\phi^{d/u}(x, y)$ and $R^{u/d}$ in terms of $R^{d/u}$ via connection potentials $\psi^{d/u}(x, y)$ which we now set out to calculate. We first need to establish the functions $f^{d/u}(y)$ defining the boundary condition (3.6) and derived from solutions of (3.5) and satisfying either (2.3) or (2.4) where appropriate. Thus we find

$$f^d(y) = \begin{cases} \cosh ly - (K/l) \sinh ly, & l \neq 0 \\ 1 - Ky, & l = 0 \end{cases} \quad (5.10)$$

and

$$f^u(y) = \begin{cases} \cosh l(h - y), & l \neq 0 \\ 1, & l = 0. \end{cases} \quad (5.11)$$

Now $\psi^{d/u}(x, y)$ satisfies (2.2)–(2.4) with (3.2), (3.6) with either (5.10) or (5.11) and (3.8). We write a general series expansion

$$\psi^{d/u}(x, y) = \tilde{R}^{d/u} e^{i\alpha x} \varphi_0(y) + \sum_{n=1}^{\infty} \tilde{A}_n^{d/u} e^{-\alpha_n x} \varphi_n(y) \quad (5.12)$$

and follow the solution process outlined above for the scattering problems, so that

$$\tilde{R}^{d/u} = -\frac{i}{\alpha h N_0} \int_{G_{d/u}} \psi_x^{d/u}(0, y) \varphi_0(y) dy \quad (5.13)$$

and

$$\tilde{A}_n^{d/u} = -\frac{1}{\alpha_n h N_n} \int_{G_{d/u}} \psi_x^{d/u}(0, y) \varphi_n(y) dy \quad (5.14)$$

where $\psi_x^{d/u}(0, y)$ is treated as an unknown function, leading to the integral equation

$$\int_{G_{d/u}} \psi_x^{d/u}(0, t) K(y, t) dt = \tilde{R}^{d/u} \varphi_0(y) - f^{d/u}(y), \quad y \in G_{d/u}. \quad (5.15)$$

Linearity allows us to write

$$\psi_x^{d/u}(0, y) = \tilde{R}^{d/u} U^{d/u}(y) - \tilde{U}^{d/u}(y) \quad (5.16)$$

in terms of $U^{d/u}(y)$, the solution of (5.8) and $\tilde{U}^{d/u}(y)$ satisfying

$$\int_{G_{d/u}} \tilde{U}^{d/u}(t) K(y, t) dt = f^{d/u}(y), \quad y \in G_{d/u} \quad (5.17)$$

which only differs from (5.8) in the right-hand side function. Using (5.16) in (5.13) gives

$$\tilde{R}^{d/u} = -i \tilde{R}^{d/u} \mathcal{B}^{d/u} + i \tilde{\mathcal{B}}^{d/u} \quad (5.18)$$

after invoking (5.9) and defining

$$\tilde{\mathcal{B}}^{d/u} = \frac{1}{\alpha h N_0} \int_{G_{d/u}} \tilde{U}^{d/u}(t) \varphi_0(y) dy. \quad (5.19)$$

Finally (5.18) can be expressed using (5.7) as

$$\tilde{R}^{d/u} = i R^{d/u} \tilde{\mathcal{B}}^{d/u} \quad (5.20)$$

and this allows (3.9) to be rearranged into the form

$$R^{u/d} = 1 - R^{d/u} \left(1 + i A^{d/u} \tilde{\mathcal{B}}^{d/u} \right) \quad (5.21)$$

which connects the reflection coefficient from one barrier problem to that of the complementary barrier problem. The constant $A^{d/u}$ is defined by the condition (3.7) which is translated here using (5.5) and (5.16) into the condition

$$\lim_{y \rightarrow a} \left(R^{d/u} U^{d/u}(y) + A^{d/u} (\tilde{R}^{d/u} U^{d/u}(y) - \tilde{U}^{d/u}(y)) \right) = 0. \quad (5.22)$$

This needs to be calculated numerically and this is outlined, along with the method for numerically approximating $\tilde{\mathcal{B}}^{d/u}$, in Appendix B.

$a/h = \frac{1}{4}, \theta = 30^\circ, ka = \frac{1}{2}$		
P	R^d (formula)	R^d (direct)
1	$0.051449 - 0.220912i$	$0.067944 - 0.251649i$
2	$0.067236 - 0.250460i$	$0.067957 - 0.251672i$
4	$0.067951 - 0.251662i$	$0.067957 - 0.251672i$
8	$0.067952 - 0.251664i$	$0.067957 - 0.251672i$
$a/h = \frac{1}{2}, \theta = 0^\circ, ka = 1$		
P	R^d (formula)	R^d (direct)
1	$0.012175 - 0.109668i$	$0.013338 - 0.114721i$
2	$0.013321 - 0.114647i$	$0.013338 - 0.114721i$
4	$0.013337 - 0.114717i$	$0.013338 - 0.114721i$
8	$0.013337 - 0.114718i$	$0.013338 - 0.114721i$

Table 1 Comparison of convergence of the complex reflection coefficient with numerical truncation parameter P using the connection formula and a direct computation.

5.1 Numerical results

Appendix B summarises the method used for approximating solutions of the integral equations numerically using the efficient and accurate approach presented by Porter and Evans (11).

Two sets of typical results are presented in table 1 where we show the complex reflection coefficient R^d for a bottom-mounted barrier extending from $y = h$ to $y = a$ computed using two different methods. In the right-hand column of each set of results we show R^d computed directly using the methods described in Porter and Evans (11) for bottom-mounted barriers. The values of R^d converge rapidly with in the number of terms, $P + 1$, in the series expansion – see (B.1). Shown in the left-hand columns are values of R^d computed using the formula (5.21) involving R^u and properties of the solution of the connection potential. Computational details are given in Appendix B. These results are also dependent on a numerical truncation parameter P and, although convergence is less rapid, the results are clearly tending to those made from the direct calculations. Results for $P = 0$ (a one-term approximation) are not shown since it can be shown analytically that the estimate for R^d with $P = 0$ using the formula (5.21) is always identically zero.

Numerical experiments suggest that the values $|R^d|$ computed using the formula (5.21) via a Galerkin method are bounded above by their exact values although we have not been able to prove the existence of bounds.

5.2 Remarks

In Porter and Evans (11) two integral equation formulations were used to provide bounds on reflection coefficients. Here it has been sufficient to present just one formulation which has been based on an unknown function $U^{u/d}(y)$ related to the velocity across the gap $G_{u/d}$ in the barrier. The alternative formulation demonstrated in Porter and Evans (11) is based on a function related to the unknown pressure, $P^{u/d}(y)$ say, on the barrier $B_{u/d}$. A similar comment was made in relation to the case of infinite depth in the previous section. An immediate advantage of this is that it allows us to calculate $R^{u/d}$ or $R^{u/d}$ via two independent methods.

The availability of dual formulations (referred to as ‘complementary formulations’ in Porter and Evans (11)) of the problems also provide an alternative insight into the connection between the

solutions to complementary problems of barriers and gaps. Thus, pursuing an integral equation formulation for a suitably defined function $P^{u/d}(y)$ leads to the integro-differential equation

$$\frac{1}{\alpha^2} \left(l^2 - \frac{d^2}{dy^2} \right) \int_{B_{u/d}} P^{u/d}(t) K(y, t) dt = \varphi_0(y), \quad y \in B_{u/d} \quad (5.23)$$

where, remarkably, $K(y, t)$ is the same kernel (5.9) defined in the integral equation (5.8) for $U^{u/d}(y)$ when $y \in G_{u/d}$ (that is over the interval complementary to that over which (5.23) is defined.) Since $G_{u/d} = B_{d/u}$ we may simultaneously reverse the ordering of u and d in (5.23), interchange $B_{u/d}$ with $G_{d/u}$ and integrate up the differential operator, respecting boundary conditions on $y = 0$ and $y = h$ and find that

$$\int_{G_{u/d}} P^{d/u}(t) K(y, t) dt = -\varphi_0(y) + C^{u/d} f^{u/d}(y), \quad y \in G_{u/d}. \quad (5.24)$$

The right-hand side involves a constant of integration $C^{u/d}$ (not the same as $A^{d/u}$ previously in (3.1)) and $f^{u/d}(y)$ is precisely the function defined by (3.6) in the specification of the connection potential. It follows that

$$P^{d/u}(t) = -U^{u/d}(t) + C^{u/d} \tilde{U}^{u/d}(t), \quad t \in G_{u/d} \quad (5.25)$$

since $U^{u/d}(t)$ and $\tilde{U}^{u/d}(t)$ are solutions to (5.8) and (5.17).

Thus (5.25) indicates that a property of the solution to a d -problem has been related to properties of solutions to complementary u -problems and vice versa.

The approach outlined above, which can be developed further and applies also to the infinite depth case of the previous section, illustrates that the integral equation formulations themselves may be used to connect complementary barrier problems. The key feature which allows this to happen is the structure of (5.23) and, in particular, the presence of $K(y, t)$ in the integral operator of both (5.8) and (5.23).

6. Conclusion

In this paper it has been shown that the solution to a particular problem of surface wave scattering by a thin vertical barrier with a gap can be related to the complementary problem where the gaps and the barriers are interchanged. This is done by using a solution to an auxiliary problem involving the interchanged gap/barrier arrangement in which a non-physical forcing replaces the incident wave forcing at the barrier. We have demonstrated analytically that the explicit solution to the Ursell (2) problem for a surface-piercing barrier can be found from the explicit Dean (1) submerged barrier solution and its associated auxiliary problem, solved using Dean-type methods. In the latter part of the paper we have also shown how numerical solutions demonstrate the connection between the two complementary problems in the case of finite depth and oblique wave incidence.

This connection result appears to be mainly of theoretical interest and has little obvious practical significance. In the examples given, explicit solutions (or solution methods) exist to each of the complementary problems and we have simply confirmed that the theory connecting the complementary problems works as it should. Had it been, for example, that the Ursell (2) explicit solution was not known – but the Dean (1) solution known – then the theory presented here would have allowed us to find the Ursell result.

However, there may be other examples in water waves or other linear field theories where similar ideas to those developed here (which themselves are really adaptations of Babinet's principle) can be used to generate solutions to new problems.

Within this paper we have only considered a single barrier and a single gap and this eases the presentation. The extension of the theory connecting problems involving complementary arrangements of multiple barriers and/or gaps is not difficult. It is found that as many connection potentials are needed as there are submerged barrier edges (just one in our examples), each one defined by forcing on a single section of the barrier.

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APPENDIX A

Integral results

We start with the result

$$\int_0^\infty \frac{J_0(ka) \sin ky}{k^2 + K^2} dk = \frac{\sinh Ky}{K} K_0(Ka), \quad y \in (0, a) \quad (\text{A.1})$$

(see Mandal and Chakrabarti, (9, p.106 equation 23)). Thus

$$\mathcal{D}_y^- \int_0^\infty \frac{J_0(ka) \sin ky}{k^2 + K^2} dk = \int_0^\infty \frac{J_0(ka) L(k, y)}{k^2 + K^2} dk = e^{-Ky} K_0(Ka). \quad (\text{A.2})$$

We also have

$$\int_0^\infty \frac{J_2(ka) \sin ky}{k^2 + K^2} dk = \frac{2}{K^2 a} \left(\frac{y}{a} - K_1(Ka) \sinh Ky \right) - \frac{1}{K} K_0(Ka) \sinh Ky, \quad y \in (0, a), \quad (\text{A.3})$$

another result quoted in Mandal and Chakrabarti (9, p.106 equation 17). It follows that

$$\begin{aligned} \mathcal{D}_y^- \left(\int_0^\infty \frac{J_2(ka) \sin ky}{k^2 + K^2} dk \right) &= \int_0^\infty \frac{J_2(ka) L(k, y)}{k^2 + K^2} dk \\ &= \frac{2}{Ka} \left(\frac{1 - Ky}{Ka} - K_1(Ka) e^{-Ky} \right) \end{aligned} \quad (\text{A.4})$$

for $y \in (0, a)$ so that from (A.2) and (A.4) and using $2J_1(z) = z(J_2(z) + J_0(z))$ we have

$$\begin{aligned} \frac{a}{2} \int_0^\infty \frac{(J_2(ka) + J_0(ka)) L(k, y)}{k^2 + K^2} dk &= \int_0^\infty \frac{J_1(ka) L(k, y)}{k(k^2 + K^2)} dk \\ &= \frac{1}{K} \left(\frac{1 - Ky}{Ka} - K_1(Ka) e^{-Ky} \right). \end{aligned} \quad (\text{A.5})$$

APPENDIX B

Galerkin approximation to solution of integral equations

The numerical solution to the integral equation (5.8) is outlined in Porter and Evans (11), although the addition of new integral equation (5.17) associated with the connection potential merits repeating some of the outline details here. For the two integral equations we make the $(P + 1)$ -term series expansions

$$U(y) \approx \sum_{p=0}^P c_p v_p(y) \quad \text{and} \quad \tilde{U}(y) \approx \sum_{p=0}^P \tilde{c}_p v_p(y), \quad y \in G \quad (\text{B.1})$$

(dropping the u/d super/subscripts for clarity) where $v_p(y)$ are functions carefully chosen to reflect the physical properties of the problem and c_p, \tilde{c}_p are coefficients to be found. Then the Galerkin method applied to (5.8) and (5.17) results in the algebraic system of equations

$$\sum_{p=0}^P c_p K_{pq} = F_q, \quad \text{and} \quad \sum_{p=0}^P \tilde{c}_p K_{pq} = \tilde{F}_q, \quad q = 0, 1, \dots, P, \quad (\text{B.2})$$

where

$$K_{pq} = \sum_{n=1}^{\infty} \frac{F_{pn} F_{qn}}{\alpha_n h N_n} \quad \text{and} \quad F_{pn} = \int_G v_p(y) \varphi_n(y) dy \quad (\text{B.3})$$

for $p = 0, 1, \dots$ and $n = 0, 1, 2, \dots$. In (B.2) $F_p = F_{p0}$ whilst

$$\tilde{F}_p = \int_G v_p(z) f(y) dy \quad p = 0, 1, \dots \quad (\text{B.4})$$

Then using (B.1) in (5.10) and (5.20) with (B.3) gives

$$\mathcal{B} \approx \frac{1}{\alpha h N_0} \sum_{p=0}^P c_p F_{p0}, \quad \text{and} \quad \tilde{\mathcal{B}} \approx \frac{1}{\alpha h N_0} \sum_{p=0}^P \tilde{c}_p F_{p0}, \quad (\text{B.5})$$

which allows R and \tilde{R} to be approximated using (5.7) and (5.20).

The specification of $v_p(y)$ depends on where the gap is located and different cases are discussed in detail

in Porter and Evans (11). To illustrate the results, we choose the simplest of those cases here where the gap is defined as $G = G_u = (a, h)$ (implying super/subscripts u apply throughout) and define

$$v_p(y) = \frac{2(-1)^p}{\pi((h-a)^2 - (h-y)^2)^{1/2}} T_{2p}\left(\frac{h-y}{h-a}\right) \quad (\text{B.6})$$

in terms of even Chebychev polynomials, $T_{2p}(z)$. It follows, from Porter and Evans (11), that

$$F_{pn} = J_{2p}(k_n(h-a)), \quad n \geq 1, \quad \text{and} \quad F_{p0} = (-1)^p I_{2p}(k(h-a)) \quad (\text{B.7})$$

whilst, with $f(y) = f^u(y)$ given by (5.11),

$$\tilde{F}_p = \begin{cases} (-1)^p I_{2p}(l(h-a)), & l \neq 0, \\ \delta_{p0}, & l = 0. \end{cases} \quad (\text{B.8})$$

The final part of the numerical procedure is to determine the constant A^u from the condition (5.22). Using (B.1) in (5.22) and noting that $T_{2p}(1) = 1$, (5.22) reduces numerically into solving this relation for A^u :

$$A^u \left(\tilde{R}^u \sum_{p=0}^P (-1)^p c_p - \sum_{p=0}^P (-1)^p \tilde{c}_p \right) \approx -R^u \sum_{p=0}^P (-1)^p c_p. \quad (\text{B.9})$$